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$\mathcal{N} = 4$ SYM on $R \times S^3$ and PP-Wave

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Abstract: We consider the radial quantization of $\mathcal{N} = 4$ super Yang-Mills (SYM) in 4 dimensions, i.e., $\mathcal{N} = 4$ SYM on a cylinder $\mathbf{R} \times \mathbf{S}^3$. We construct the generators of superconformal symmetry in the case of U(N) gauge group, generalizing the earlier work by Nicolai et al. for U(1) gauge group. We study how these generators contract to the symmetry of pp-wave when they act on a state with large R-charge.

Keywords: Penrose limit and pp-wave background, Extended Supersymmetry, AdS-CFT and dS-CFT Correspondence.

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1. Introduction

Recently, Berenstein, Maldacena and Nastase proposed that the type-IIB string theory on a pp-wave background is dual to a sector of $\mathcal{N}=4$ SYM with large R-charge [1]. On the bulk side, the pp-wave geometry is obtained from $AdS_5\times\mathbf{S}^5$ by focusing on a null geodesic rotating around the equator of $\mathbf{S}^5$ [2, 1]. The important fact here is that the Green-Schwarz string on this background is solvable in the lightcone gauge [3]. On the YM side, this limit corresponds to taking a double scaling limit [1, 4, 5]

$$N, J\to\infty\,,\qquad g_2=\frac{J^2}{N},\ \ \lambda'=\frac{g_{\mathrm{YM}}^2 N}{J^2}\quad :\text{fixed}\,. \tag{1.1}$$

One of the evidence of this duality is that the free string spectrum is reproduced from the computation of anomalous dimensions of the so-called BMN operators in $\mathcal{N}=4$ SYM [1, 4, 6, 7]. However, the precise mapping between these two sides is yet unknown.

Some proposals about the holography in the pp-wave background are addressed in [8]–[11]. Before taking the Penrose limit, the boundary of AdS_5 in the global coordinate is $\mathbf{R}\times\mathbf{S}^3$. Since the pp-wave geometry is obtained as the limit of AdS_5 with global coordinate, it is natural to think that the dual theory is a limit of $\mathcal{N}=4$ SYM on the cylinder $\mathbf{R}\times\mathbf{S}^3$. In [11], it was shown that the boundary of the pp-wave geometry is a one-dimensional null line and was proposed that the dual theory is a matrix quantum mechanics obtained by the KK reduction of $\mathcal{N}=4$ SYM on $\mathbf{S}^3$. Note that this matrix model appeared in [12, 13] in the context of giant gravitons (see also [14]). Matrix string theories with $\mathrm{U}(J)$ gauge group are discussed in [15, 16, 17].

In this paper, we study $\mathrm{U}(N)$ $\mathcal{N}=4$ SYM on $\mathbf{R}\times\mathbf{S}^3$ and some of its properties under the limit (1.1). We construct the generators of superconformal symmetry of this theory, by generalizing the earlier work for the U(1) gauge group [18]. Since $\mathbf{R}\times\mathbf{S}^3$ is a curved manifold, the superconformal symmetry is realized in a nontrivial way.

We also study the contraction of the conformal symmetry to the symmetry of pp-wave background [19, 20] from the YM viewpoint. Recently, this problem was studied in the duality between a $D=2$ CFT and a 6-dimensional pp-wave geometry [21]. The higher dimensional case is also mentioned in [21]. We explicitly perform this contraction in the free field limit of $\mathcal{N}=4$ SYM.

This paper is organized as follows. In section 2, we construct the generators of conformal symmetry of $\mathcal{N}=4$ SYM on $\mathbf{R}\times\mathbf{S}^3$. In section 3, we construct the conformal Killing spinor on $\mathbf{R}\times\mathbf{S}^3$ and the generators of superconformal symmetry. In section 4, we summarize the KK spectrum of $\mathcal{N}=4$ SYM on $\mathbf{S}^3$ and study its pp-wave limit. In section 5, we study the contraction of the conformal symmetry to the symmetry of pp-wave geometry. Section 6 is devoted to discussions. In appendix A, we summarize our notation of Γ-matrices. Appendix B is the list of conformal Killing vectors on $\mathbf{R}\times\mathbf{S}^3$.

2. Conformal symmetry of $\mathcal{N}=4$ SYM on $\mathbf{R}\times\mathbf{S}^3$

In this section, we construct the generators of conformal symmetry of $\mathcal{N}=4$ SYM on a cylinder. To make this paper self-contained, we review some basic facts about $\mathcal{N}=4$ SYM on $\mathbf{R}\times\mathbf{S}^3$ [18, 22, 23].

2.1 Conformal coupling to a background metric

$\mathcal{N}=4$ SYM can couple to a background metric in a Weyl invariant way, since this theory is conformally invariant.[1] The action can be written as

$$S[A_\mu,X_m,\lambda,g_{\mu\nu}] = -\frac{1}{g_{\mathrm{YM}}^2}\int d^4x\sqrt{g}\,\mathrm{Tr}\biggl(\frac{1}{2}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}+g^{\mu\nu}D_\mu X_m D_\nu X_m+\frac{R}{6}X_m^2- \\ - i\overline{\lambda}\Gamma^\mu D_\mu\lambda-\overline{\lambda}\Gamma^m[X_m,\lambda]-\frac{1}{2}[X_m,X_n]^2\biggr) \quad (2.1)$$

[1] For a generic background metric, there appears a Weyl anomaly written as a combination of Riemann tensor. However, this anomaly vanishes when the metric is $\mathbf{R}\times\mathbf{S}^3$ which is relevant for our discussion of the radial quantization.

where $\mu, \nu = 0, \ldots, 3$, $m, n = 4, \ldots, 9$, and D_μ is the covariant derivative including the gauge field

$$D_\mu \lambda = \nabla_\mu \lambda - i[A_\mu, \lambda] = \partial_\mu \lambda + \widetilde{A}_\mu \lambda - i[A_\mu, \lambda] \,. \tag{2.2}$$

$\widetilde{A}_\mu$ is the spin connection written as $\widetilde{A}_\mu = \frac{1}{4}\Omega_\mu^{ab}\Gamma_{ab}$. Here Ω^{ab} is the connection 1-form defined by $d\omega^a + \Omega^a{}_b \wedge \omega^b = 0$, and ω^a is the vierbein defined in the usual manner:

$$g_{\mu\nu} = \eta_{ab}\omega_\mu^a \omega_\nu^b \,, \qquad g^{\mu\nu} = \eta^{ab} e_a^\mu e_b^\nu \,, \qquad \{\Gamma^\mu, \Gamma^\nu\} = 2g^{\mu\nu} \,, \qquad \{\Gamma^a, \Gamma^b\} = 2\eta^{ab} \,. \tag{2.3}$$

One can show that the action (2.1) is Weyl invariant

$$S[A_\mu, X_m, \lambda, g_{\mu\nu}] = S[A_\mu, e^{-\alpha}X_m, e^{-\frac{3}{2}\alpha}\lambda, e^{2\alpha}g_{\mu\nu}] \,, \tag{2.4}$$

by noting that the scalar curvature and the spin connection in n-dimension transform under the Weyl rescaling as

$$\begin{aligned} e^{2\alpha}R(e^{2\alpha}g_{\mu\nu}) &= R(g_{\mu\nu}) - 2(n-1)g^{\mu\nu}\nabla_\mu\partial_\nu\alpha - (n-1)(n-2)g^{\mu\nu}\partial_\mu\alpha\partial_\nu\alpha \\ \widetilde{A}_\mu(e^{2\alpha}g_{\mu\nu}) &= \widetilde{A}_\mu(g_{\mu\nu}) + \frac{1}{2}\gamma_{\mu\nu}\partial^\nu\alpha \,. \end{aligned} \tag{2.5}$$

The metric on the plane $\mathbf{R}^4$ and the cylinder $\mathbf{R} \times \mathbf{S}^3$ are related by a Weyl rescaling

$$ds^2 = dr^2 + r^2 d\Omega_3^2 = e^{2\tau}(d\tau^2 + d\Omega_3^2) \tag{2.6}$$

where τ and r are related by

$$\tau = \log r \,. \tag{2.7}$$

Therefore, the dilatation on $\mathbf{R}^4$ corresponds to the time translation on $\mathbf{R} \times \mathbf{S}^3$ after the Wick rotation $\tau = it$, and the hamiltonian H on the cylinder can be identified with the weight Δ on the plane.

Now let us fix $g_{\mu\nu}$ to be the metric on $\mathbf{R} \times \mathbf{S}^3$

$$ds^2 = -dt^2 + d\Omega_3^2 = -dt^2 + d\theta^2 + \sin^2\theta(d\psi^2 + \sin^2\psi d\chi^2) \,. \tag{2.8}$$

Since $\mathbf{R} \times \mathbf{S}^3$ has the scalar curvature $R = 6$, the action (2.1) becomes

$$\begin{aligned} S &= \frac{2}{g_{\rm YM}^2}\int d^4x \,\mathrm{Tr}\left(-\frac{1}{4}F_{MN}^2 + \frac{i}{2}\bar\lambda\Gamma^M D_M\lambda - \frac{1}{2}X_m^2\right) \\ &= \frac{2}{g_{\rm YM}^2}\int d^4x \,\mathrm{Tr}\bigg(-\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}(D_\mu X^m)^2 + \frac{1}{4}[X_m, X_n]^2 + \\ &\qquad + \frac{i}{2}\bar\lambda\Gamma^\mu D_\mu\lambda + \frac{1}{2}\bar\lambda\Gamma^m[X^m, \lambda] - \frac{1}{2}X_m^2\bigg) \end{aligned} \tag{2.9}$$

where $A_M = (A_\mu, X_m)$ with $M, N = 0, \ldots, 9$. In (2.1) and (2.9), λ is a 10-dimensional Majorana-Weyl spinor dimensionally reduced to 4 dimensions. Note that there is no Coulomb branch in this theory because of the mass term of the scalar fields X_m.

In SU(4) symmetric notation, this action is written as

$$S = \frac{2}{g_{\rm YM}^2}\int d^4x \mathrm{Tr}\Big\{ -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}D_\mu X_{AB}D^\mu X^{AB} + \frac{1}{4}[X_{AB}, X_{CD}][X^{AB}, X^{CD}] + \\ + i\overline{\lambda}_{+A}\not{D}\lambda_+^A + \overline{\lambda}_-^A[X_{AB}, \lambda_+^B] + \overline{\lambda}_{+A}[X^{AB}, \lambda_{-B}] - \frac{1}{2}X_{AB}X^{AB}\Big\}, \tag{2.10}$$

where the gaugino λ_+^A is a $D = 4$ positive chirality Weyl spinor and transforms as **4** of $\mathrm{SU}(4)_R$ symmetry, and the scalars $X^{AB} = -X^{BA}$ are **6** of $\mathrm{SU}(4)_R$. $\not{D}$ means $\gamma^\mu D_\mu$. See appendix A for our notations.

2.2 Conformal Killing vector on $\mathbf{R} \times \mathbf{S}^3$

After fixing the metric (2.8), the Weyl rescaling (2.4) is no longer a symmetry of the action. Instead, the conformal symmetry is generated by the conformal Killing vectors of this fixed metric (2.8). One easy way to find the conformal Killing vectors on $\mathbf{R} \times \mathbf{S}^3$ is to take the limit of the Killing vectors on AdS_5. To write the Killing vectors on AdS_5, it is convenient to regard AdS_5 as a hyperboloid in $\mathbf{R}^{4,2}$:

$$\sum_{a,b=0}^{5} \widehat{\eta}_{ab}y^a y^b = y_0^2 - \sum_{a=1}^{4} y_a^2 + y_5^2 = R^2 \tag{2.11}$$

where $\widehat{\eta}_{ab} = \mathrm{diag}(+, (-)^4, +)$. In terms of the coordinate y^a, the Killing vectors on AdS_5 can be easily written as

$$L_{ab} = y_a\frac{\partial}{\partial y^b} - y_b\frac{\partial}{\partial y^a}\,. \tag{2.12}$$

To obtain the conformal Killing vectors on $\mathbf{R} \times \mathbf{S}^3$, we have to rewrite L_{ab} in terms of the global coordinate (ρ, t, Ω_3), in which the metric of AdS_5 takes the form

$$ds^2 = R^2\left(-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\Omega_3^2\right)\,. \tag{2.13}$$

This coordinate and the coordinate y^a in (2.11) are related by

$$y^0 + iy^5 = R\,e^{it}\cosh\rho\,, \qquad y^a = R\,n^a\sinh\rho \qquad (a = 1, \ldots, 4) \tag{2.14}$$

where n^a is a unit vector on $\mathbf{S}^3$. In our parameterization of $\mathbf{S}^3$ (2.8), n^a is given by

$$n = (\cos\theta, \sin\theta\cos\psi, \sin\theta\sin\psi\cos\chi, \sin\theta\sin\psi\sin\chi)\,. \tag{2.15}$$

Now the (conformal) Killing vectors ξ_{ab} on $\mathbf{R} \times \mathbf{S}^3$ can be obtained as the value of L_{ab} on the boundary of AdS_5:

$$\lim_{\rho\to\infty} L_{ab} = \xi_{ab}\,. \tag{2.16}$$

They satisfy the SO(4, 2) algebra

$$[\xi_{ab}, \xi_{cd}] = \widehat{\eta}_{ad}\xi_{bc} - \widehat{\eta}_{bd}\xi_{ac} - \widehat{\eta}_{ac}\xi_{bd} + \widehat{\eta}_{bc}\xi_{ad}\,. \tag{2.17}$$

On $\mathbf{R}\times\mathbf{S}^3$, there are seven Killing vectors which are the generators of the $\mathbf{R}\times\mathrm{SO}(4)$ isometry of the metric (2.8):

$$\xi_{05}=\partial_t\,,\qquad \xi_{ab}=-n^a\partial_b+n^b\partial_a\qquad (a,b=1,\dots,4)\,. \tag{2.18}$$

Note that the corresponding L_{ab}'s are independent of ρ. In addition to these Killing vectors, there are eight conformal Killing vectors on $\mathbf{R}\times\mathbf{S}^3$. On AdS_5, they are written as

$$L_{0a}-iL_{5a}=e^{-it}[n^a(-i\tanh\rho\,\partial_t+2\partial_\rho)+\coth\rho\,\partial_a]\qquad (a=1,\dots,4)\,. \tag{2.19}$$

Taking the limit $\rho\to\infty$, we get

$$\xi_{0a}-i\xi_{5a}=e^{-it}(-in^a\partial_t+\partial_a)\,. \tag{2.20}$$

∂_a in (2.18) and (2.20) is defined by

$$\partial_a=\frac{\partial_\theta n^a}{(\partial_\theta n)^2}\partial_\theta+\frac{\partial_\psi n^a}{(\partial_\psi n)^2}\partial_\psi+\frac{\partial_\chi n^a}{(\partial_\chi n)^2}\partial_\chi\,,\qquad \partial_a n^b=\delta^{ab}-n^a n^b\,. \tag{2.21}$$

One can check that the vectors (2.20) satisfy the conformal Killing equation

$$\nabla_\mu\xi_{\nu ab}+\nabla_\nu\xi_{\mu ab}=2\Omega_{ab}g_{\mu\nu} \tag{2.22}$$

with

$$\Omega_{0a}-i\Omega_{5a}=-e^{-it}n^a\,. \tag{2.23}$$

In appendix B, we write down the explicit form of ξ_{ab}.

2.3 Generators of conformal symmetry

The action (2.10) is invariant under the $\mathrm{SO}(4,2)$ conformal transformation generated by the conformal Killing vectors ξ_{ab}

$$\begin{aligned}\delta_{\xi_{ab}}A_\mu &= \xi^\nu_{ab}\nabla_\nu A_\mu+\nabla_\mu\xi^\nu_{ab}A_\nu \\ \delta_{\xi_{ab}}X^{AB} &= \xi^\mu_{ab}\partial_\mu X^{AB}+\Omega_{ab}X^{AB}\delta_{\xi_{ab}}\lambda^A_+=\xi^\mu_{ab}\nabla_\mu\lambda^A_++\frac{1}{4}\nabla_\mu\xi_{\nu ab}\gamma^{\mu\nu}\lambda^A_++\frac{3}{2}\Omega_{ab}\lambda^A_+\,.\end{aligned} \tag{2.24}$$

The terms including ξ_{ab} represent the Lie derivative along ξ_{ab} and the coefficient in front of Ω_{ab} is determined by the weight of the field. To check this invariance, we need an equation following from the definition of Ω_{ab} (2.22) [24]

$$(n-1)\nabla^\mu\partial_\mu\Omega_{ab}+R\Omega_{ab}+\frac{1}{2}\xi^\mu_{ab}\partial_\mu R=0\,. \tag{2.25}$$

In the case of $\mathbf{R}\times\mathbf{S}^3$, this reads

$$\nabla^\mu\partial_\mu\Omega_{ab}+2\Omega_{ab}=0\,. \tag{2.26}$$

The Noether charges of this symmetry (2.24) are found to be

$$M_{ab} = \frac{2}{g_{\rm YM}^2}\mathrm{Tr}\int_{\mathbf{S}^3}\Bigg[\xi^0_{ab}\Bigg(\frac{1}{2}D_\mu X_{AB}D^\mu X^{AB}+\frac{1}{2}X_{AB}X^{AB}+\frac{1}{4}F^2_{\mu\nu}-i\overline{\lambda}_{+A}\not{D}\lambda^A_+ - \tag{2.27}$$
$$-\frac{1}{4}[X_{AB},X_{CD}][X^{AB},X^{CD}]-\overline{\lambda}^A_-[X_{AB},\lambda^B_+]-\overline{\lambda}_{+A}[X^{AB},\lambda_{-B}]\Bigg)+$$
$$+\xi^\mu_{ab}(D_0X_{AB}D_\mu X^{AB}+F_{0\nu}F_\mu{}^\nu)+i\overline{\lambda}_{+A}\gamma^0\delta_{\xi_{ab}}\lambda^A_+ +\frac{1}{2}\Omega_{ab}\overleftrightarrow{\partial}_0(X_{AB}X^{AB})\Bigg].$$

Note that the hamiltonian H is given by M_{05}.

2.4 $\mathrm{SU}(4)_R$ symmetry

The action (2.10) is also invariant under the SU(4) R-symmetry

$$\delta\lambda^A_+ = iT^A{}_B\lambda^B_+\,,\qquad \delta\overline{\lambda}_{-A} = -i\overline{\lambda}_{-B}T^B{}_{-A}\,,\qquad \delta X^{AB} = iT^A{}_C X^{CB} + iT^B{}_C X^{AC}\,, \tag{2.28}$$

where $T^A{}_B$ is a hermitian traceless matrix. The charge of this symmetry is

$$J^A{}_B = \frac{2}{g_{\rm YM}^2}\mathrm{Tr}\int_{\mathbf{S}^3}\Big(-2iX^{AC}D_0X_{CB} - \overline{\lambda}_{+B}\gamma^0\lambda^A_+\Big)\,. \tag{2.29}$$

In the SO(6) notation, this is written as

$$J^{mn} = \frac{i}{2}(\Gamma^{mn})^A{}_B J^B{}_A = \frac{2}{g_{\rm YM}^2}\mathrm{Tr}\int_{\mathbf{S}^3}\Big(X^m\overleftrightarrow{D}_0X^n - \frac{i}{4}\overline{\lambda}\Gamma^0\Gamma^{mn}\lambda\Big)\,. \tag{2.30}$$

3. Superconformal symmetry of $\mathcal{N}=4$ SYM on $\mathbf{R}\times\mathbf{S}^3$

3.1 Conformal Killing spinor on $\mathbf{R}\times\mathbf{S}^3$

The superconformal symmetry is generated by the conformal Killing spinors on $\mathbf{R}\times\mathbf{S}^3$. They can be obtained from the Killing spinors on AdS_5 defined by

$$\widetilde{\nabla}_\mu\epsilon = \left(\nabla_\mu - \frac{1}{2}\gamma_\mu\right)\epsilon = 0\,. \tag{3.1}$$

In the global coordinate (2.13), $\widetilde{\nabla}$ is written as

$$\begin{aligned}
\widetilde{\nabla}_0 &= \partial_0 + \frac{1}{2}\sinh\rho\gamma_0\gamma_\rho - \frac{1}{2}\cosh\rho\gamma_0 = e^{-\frac{1}{2}\rho\gamma_5}\left(\partial_0 - \frac{1}{2}\gamma_0\right)e^{\frac{1}{2}\rho\gamma_5}\\
\widetilde{\nabla}_i &= \nabla_i + \frac{1}{2}\cosh\rho\gamma_i\gamma_\rho - \frac{1}{2}\sinh\rho\gamma_i = e^{-\frac{1}{2}\rho\gamma_5}\left(\nabla_i - \frac{1}{2}\gamma_i\gamma_5\right)e^{\frac{1}{2}\rho\gamma_5}\\
\widetilde{\nabla}_\rho &= \partial_\rho - \frac{1}{2}\gamma_\rho = \partial_\rho + \frac{1}{2}\gamma_5\,,
\end{aligned}\tag{3.2}$$

where $i(=1,2,3)$ denotes the direction of $\mathbf{S}^3$ and we identified $\gamma_\rho = -\gamma_5$. Therefore, the Killing spinor on AdS_5 and the conformal Killing spinor on $\mathbf{R}\times\mathbf{S}^3$ are related by [22, 25]

$$\epsilon_{AdS_5} = e^{-\frac{1}{2}\rho\gamma_5}\epsilon_{\mathbf{R}\times\mathbf{S}^3}\,, \tag{3.3}$$

where $\epsilon_{\mathbf{R}\times\mathbf{S}^3}$ obeys

$$\partial_0 \epsilon = \frac{1}{2}\gamma_0 \epsilon\,, \qquad \nabla_i \epsilon = \frac{1}{2}\gamma_i \gamma_5 \epsilon\,. \tag{3.4}$$

Under the chiral decomposition $\gamma_5 \epsilon_\pm = \pm\epsilon_\pm$, (3.4) becomes

$$\nabla_\mu \epsilon_- = \frac{1}{2}\gamma_\mu \epsilon_+\,, \qquad \nabla_\mu \epsilon_+ = \frac{1}{2}\widetilde{\gamma}_\mu \epsilon_- \tag{3.5}$$

where $\widetilde{\gamma}_\mu = (\gamma_0, -\gamma_i)$. Note that this equation is consistent with the curvature of $\mathbf{R}\times\mathbf{S}^3$.

By using the connection 1-form on $\mathbf{S}^3$

$$\Omega^{12} = -\cos\theta d\psi\,, \quad \Omega^{23} = -\cos\psi d\chi\,, \quad \Omega^{31} = \cos\theta\sin\psi d\chi\,, \tag{3.6}$$

with respect to the dreibein $(\omega^1, \omega^2, \omega^3) = (d\theta, \sin\theta d\psi, \sin\theta\sin\psi d\chi)$, the solution to the conformal Killing spinor equation (3.4) is found to be

$$\epsilon = e^{\frac{t}{2}\gamma_0} e^{\frac{\theta}{2}\gamma_{15}} e^{\frac{\psi}{2}\gamma_{12}} e^{\frac{\chi}{2}\gamma_{23}} \epsilon_0\,, \tag{3.7}$$

where ϵ_0 is a constant spinor.

Note that the bilinear combination of two conformal Killing spinors $\xi_\mu = \overline{\epsilon_-^{(1)}}\gamma_\mu \epsilon_-^{(2)}$ is a conformal Killing vector [24]

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = \left(\overline{\epsilon_-^{(1)}}\epsilon_+^{(2)} - \overline{\epsilon_+^{(1)}}\epsilon_-^{(2)}\right) g_{\mu\nu}\,. \tag{3.8}$$

3.2 Superconformal transformation

The superconformal symmetry is generated by the conformal Killing spinors. In $D = 10$ notation, the transformation law is written as

$$\delta_\epsilon A_M = -i\overline{\lambda}\Gamma_M \epsilon\,, \qquad \delta_\epsilon \lambda = \frac{1}{2}F_{MN}\Gamma^{MN}\epsilon - \frac{1}{2}X_m \Gamma^m \Gamma^\mu \nabla_\mu \epsilon\,. \tag{3.9}$$

In $D = 4$ notation, this reads

$$\begin{aligned} \delta_\epsilon A_\mu &= -i\overline{\lambda}\Gamma_\mu \epsilon\,, \qquad \delta X_m = -i\overline{\lambda}\Gamma_m \epsilon \\ \delta_\epsilon \lambda &= \left[\frac{1}{2}F_{\mu\nu}\Gamma^{\mu\nu}\epsilon - D_\mu X^m \Gamma^m \Gamma^\mu - \frac{1}{2}X_m \Gamma^m \Gamma^\mu \nabla_\mu - \frac{i}{2}[X^m, X^n]\Gamma^{mn}\right]\epsilon\,. \end{aligned} \tag{3.10}$$

The commutator of two superconformal transformation is closed up to a gauge transformation:

$$[\delta_\epsilon, \delta_\eta] = \delta_{\mathrm{SO}(4,2)}(\xi^\mu) + \delta_{\mathrm{SO}(6)}(\Lambda^{mn}) + \delta_{\mathrm{gauge}}(v) \tag{3.11}$$

where the transformation parameters are given by

$$\xi^\mu = 2i\overline{\epsilon}\Gamma^\mu \eta\,, \quad \Lambda^{mn} = \frac{i}{2}(\overline{\epsilon}\Gamma^{mn}\Gamma^\mu \nabla_\mu \eta - \overline{\eta}\Gamma^{mn}\Gamma^\mu \nabla_\mu \epsilon)\,, \quad v = -2i\overline{\epsilon}\Gamma^M \eta A_M\,. \tag{3.12}$$

In SU(4) symmetric notation, the transformation (3.9) is written as

$$\begin{aligned} \delta_\epsilon A_\mu &= -i\left(\overline{\lambda}_{+A}\gamma_\mu \epsilon_+^A - \overline{\epsilon}_{+A}\gamma_\mu \lambda_+^A\right) \\ \delta_\epsilon X^{AB} &= -i\left(-\overline{\epsilon}_-^A \lambda_+^B + \overline{\epsilon}_-^B \lambda_+^A + \epsilon^{ABCD}\overline{\lambda}_{+C}\epsilon_{-D}\right) \\ \delta_\epsilon \lambda_+^A &= \frac{1}{2}F_{\mu\nu}\gamma^{\mu\nu}\epsilon_+^A + 2D_\mu X^{AB}\gamma^\mu \epsilon_{-B} + X^{AB}\nabla\!\!\!/\,\epsilon_{-B} + 2i\left[X^{AC}, X_{CB}\right]\epsilon_+^B \\ \delta_\epsilon \lambda_{-A} &= \frac{1}{2}F_{\mu\nu}\gamma^{\mu\nu}\epsilon_{-A} + 2D_\mu X_{AB}\gamma^\mu \epsilon_+^B + X_{AB}\nabla\!\!\!/\,\epsilon_+^B + 2i\left[X_{AC}, X^{CB}\right]\epsilon_{-B}\,, \end{aligned} \tag{3.13}$$

and the Noether charge of this symmetry is found to be

$$Q_\epsilon = \frac{2}{g_{\rm YM}^2}{\rm Tr}\int_{\mathbf{S}^3}\left(i\overline{\lambda}_{+A}\gamma^0\delta_\epsilon\lambda_+^A + i\overline{\lambda}_-^A\gamma^0\delta_\epsilon\lambda_{-A}\right). \tag{3.14}$$

4. Hamiltonian formalism of $\mathcal{N}=4$ SYM

4.1 KK reduction on $\mathbf{S}^3$

In this subsection, we summarize the KK reduction of $\mathcal{N}=4$ SYM on $\mathbf{S}^3$. The mass spectrum in $(0+1)$-dimension is given by [26]

$$M_{\rm scalar} = \ell+1\,, \qquad M_{\rm fermion} = \ell+\frac{3}{2}\,, \qquad M_{\rm vector} = \ell+2 \qquad (\ell=0,1,\ldots)\,. \tag{4.1}$$

The scalar harmonics on $\mathbf{S}^3$ with angular momentum ℓ is given by the traceless symmetric polynomial of n^a with degree ℓ. $M_{\rm scalar}^2$ in (4.1) can be obtained by adding the curvature $R/6=1$ and the laplacian $-\Delta_{\mathbf{S}^3} = \ell(\ell+2)$. We normalize the spherical harmonics as

$$\frac{1}{2\pi^2}\int_{\mathbf{S}^3} Y_\ell^I Y_{\ell'}^{I'} = \delta_{\ell\ell'}\delta^{II'}\,, \tag{4.2}$$

where $2\pi^2$ is the volume of $\mathbf{S}^3$. For $\ell=1$, we find

$$Y_{\ell=1}^a = 2n^a\,, \quad \frac{1}{2\pi^2}\int_{\mathbf{S}^3} Y_{\ell=1}^a Y_{\ell=1}^b = \delta^{ab}\,. \tag{4.3}$$

Let us look at the free part of the lagrangian for scalar fields (2.9). By rescaling X^m to the canonically normalized field ϕ^m and expanding in terms of the spherical harmonics

$$X^m(t,\Omega) = \frac{g_{\rm YM}}{2\pi}\phi^m = \frac{g_{\rm YM}}{2\pi}\sum_{\ell=0}^{\infty}\phi_\ell^m(t)Y_\ell(\Omega_3)\,, \tag{4.4}$$

the free lagrangian turns out to be

$$L = {\rm Tr}\sum_{\ell=0}^{\infty}\left[\frac{1}{2}(\dot{\phi}_\ell^m)^2 - \frac{1}{2}(\ell+1)^2(\phi_\ell^m)^2\right]. \tag{4.5}$$

Then the free hamiltonian can be diagonalized as

$$H = {\rm Tr}\sum_{\ell=0}^{\infty}(\ell+1)a_\ell^{m\dagger}a_\ell^m\,. \tag{4.6}$$

Here we introduced the oscillators a_ℓ^m by

$$\phi_\ell^m = \frac{1}{\sqrt{2(\ell+1)}}\left(a_\ell^m + a_\ell^{m\dagger}\right), \qquad \dot{\phi}_\ell^m = i\sqrt{\frac{\ell+1}{2}}\left(a_\ell^{m\dagger} - a_\ell^m\right). \tag{4.7}$$

They are normalized as

$$\left[(a_\ell^m)^i_j\,,\left(a_{\ell'}^{n\dagger}\right)^k_l\right] = \delta_{\ell\ell'}\delta^{mn}\delta^i_l\delta^k_j\,, \tag{4.8}$$

where i,j,k,l are the U(N) color indices. In (4.6), we neglected the zero-point energy which is cancelled by supersymmetry. For notational simplicity, the magnetic quantum number I of Y_ℓ^I is suppressed in the above equations .

To see the mass of the fermion and the vector (4.1), it is convenient to identify $\mathbf{S}^3$ as the group manifold SU(2). Then, the metric of S^3 is written as

$$ds^2 = -\frac{1}{2}\mathrm{tr}\left(U^{-1}dU\right)^2 = -\frac{1}{2}\mathrm{tr}\left(dUU^{-1}\right)^2 = \sum_{i=1}^{3}\left(\omega_L^i\right)^2 = \sum_{i=1}^{3}\left(\omega_R^i\right)^2 , \tag{4.9}$$

where $U \in \mathrm{SU}(2)$, and ω_L^i (ω_R^i) are the left (right) invariant 1-forms

$$U^{-1}dU = i\sum_{i=1}^{3}\omega_L^i\sigma_i , \qquad dUU^{-1} = i\sum_{i=1}^{3}\omega_R^i\sigma_i . \tag{4.10}$$

Let us first consider the mass of fermions with $\ell = 0$. From the Maurer-Cartan equation $d\omega_{L,R}^i = \pm\varepsilon^{ijk}\omega_{L,R}^j\omega_{L,R}^k$, the spin connection on S^3 is found to be $\Omega_k^{ij} = \pm\varepsilon^{ijk}$. The mass term of fermion comes from the spin connection

$$\gamma^\mu\widetilde{A}_\mu = \frac{1}{4}\Omega_k^{ij}\gamma_{ijk} = \pm\frac{3}{2}\gamma_{123} . \tag{4.11}$$

Next we consider the mass of vectors with $\ell = 0$. We can see that $\omega_{L,R}^i$ are co-closed and they are eigenvectors of the laplacian on $\mathbf{S}^3$ with eigenvalue -4. Therefore, in the Coulomb gauge the gauge field on $\mathbf{S}^3$ can be expanded as

$$A(t,\Omega_3) = \sum_{i=1}^{3}\left[A_L^i(t)\omega_L^i + A_R^i(t)\omega_R^i\right] + \cdots . \tag{4.12}$$

Then the modes $A_{L,R}^i$ have mass $= 2$. Note that the KK modes of the gauge field carry $\Delta - J \geq 2$ and hence they might acquire a large anomalous dimension [1]. As for the time component of gauge field $A_{\mu=0}$, it behaves as a Lagrange multiplier of the Gauss law constraint. In passing we note that under the identification $\mathbf{S}^3 \simeq \mathrm{SU}(2)$ the scalar harmonics $Y_\ell(\Omega_3)$ is given by the matrix element $\langle j,m|U|j,m'\rangle$ with spin $j = \ell/2$. The higher spinor/vector harmonics are given by the Wigner functions on the coset SO(4)/SO(3) [27].

4.2 $\mathrm{U}(1)_J$ and rotating variable

To define the pp-wave limit, we take $\mathrm{U}(1)_J$ subgroup of $\mathrm{SU}(4)_R$ as the rotation group of X^6-X^9 plane. In other words, the generator of $\mathrm{U}(1)_J$ is given by

$$J \equiv J^{69} = \frac{1}{2}\left(J^1{}_1 + J^2{}_2 - J^3{}_3 - J^4{}_4\right) . \tag{4.13}$$

Then the fields transform under $\mathrm{U}(1)_J$ rotation as

$$\begin{aligned}
&\lambda_+^{1,2} \to e^{\frac{i}{2}\theta}\lambda_+^{1,2} , \qquad \lambda_+^{3,4} \to e^{-\frac{i}{2}\theta}\lambda_+^{3,4} \qquad \left(\lambda \to e^{-\frac{1}{2}\theta\Gamma^6\Gamma^9}\lambda\right) \\
&X^{12} = \frac{1}{2}\left(X^6 + iX^9\right) \to e^{i\theta}X^{12} , \qquad X^{34} = \frac{1}{2}\left(X^6 - iX^9\right) \to e^{-i\theta}X^{34} .
\end{aligned} \tag{4.14}$$

Therefore, the "Z" field with $J = 1$ [1] is given by

$$Z = \frac{1}{\sqrt{2}}\left(\phi^6 + i\phi^9\right) = \frac{2\sqrt{2}\pi}{g_{\mathrm{YM}}}X^{12} . \tag{4.15}$$

From (4.7), the mode of Z with angular momentum ℓ is written as

$$Z_\ell = \frac{1}{\sqrt{2(\ell+1)}}\left(A_\ell + B_\ell^\dagger\right) \tag{4.16}$$

where $A_\ell = (a_\ell^6 + i a_\ell^{9\dagger})/\sqrt{2}$ and $B_\ell = (a_\ell^6 - i a_\ell^{9\dagger})/\sqrt{2}$.

In terms of the oscillators A_ℓ and B_ℓ, J is written as

$$J = \mathrm{Tr}\sum_{\ell=0}^{\infty}\left(B_\ell^\dagger B_\ell - A_\ell^\dagger A_\ell\right). \tag{4.17}$$

In the free YM limit, $H-J$ is given by

$$H - J = \mathrm{Tr}\sum_{\ell=0}^{\infty}\left[(\ell+1)a_\ell^{s\dagger}a_\ell^s + (\ell+2)A_\ell^\dagger A_\ell + \ell B_\ell^\dagger B_\ell\right] \tag{4.18}$$

where $s = 4,5,7,8$. Here we suppressed the contribution of fermions.

The lightcone hamiltonian P^- on the bulk string side corresponds to the combination $H-J$, not to the YM hamiltonian H. This sounds puzzling because in the "modified Penrose limit" [11] the lightcone time x^+ is equal to the global time t. However, this can be reconciled by introducing the "rotating variables" by performing the time-dependent $\mathrm{U}(1)_J$ rotation

$$Z \to e^{it}Z\,, \qquad \lambda \to e^{-\frac{1}{2}t\Gamma^6\Gamma^9}\lambda\,. \tag{4.19}$$

For example, the free lagrangian of this new Z is written as

$$L = \mathrm{Tr}\sum_{\ell=0}^{\infty}\left[|\partial_t(e^{it}Z_\ell)|^2 - (\ell+1)^2|Z_\ell|^2\right]. \tag{4.20}$$

One can see that the hamiltonian H_{rot} with respect to these variables corresponds to P^-:

$$H_{\mathrm{rot}} = H - J\,. \tag{4.21}$$

Note that the replacement (4.19) corresponds to focusing on the null geodesic $\theta = t$ from the bulk string viewpoint.

From (4.18), we can see that $H-J=0$ states are generated by $B_0^\dagger$, and the creation operators $B_1^{a\dagger}$ and $a_0^{s\dagger}$ raise $H-J$ by one. Here the superscript a of $B_1^{a\dagger}$ is the magnetic index of $Y_{\ell=1}^a$. If we define the real scalar $\widetilde{\phi}_1^a$ by

$$\widetilde{\phi}_1^a = \frac{1}{\sqrt{2}}\left(B_1^a + B_1^{a\dagger}\right), \tag{4.22}$$

then $(\widetilde{\phi}_1^a, \phi_0^s)$ transforms as a vector under $\mathrm{SO}(4)\times\mathrm{SO}(4)$. Note that $\widetilde{\phi}_1^a$ corresponds to the operator $D_\mu Z$ on the plane $\mathbf{R}^4$.

The general $H-J=0$ state is written as

$$|J_1,\ldots,J_n\rangle = \prod_{i=1}^{n}\mathrm{Tr}\left(B_0^{\dagger J_i}\right)|0\rangle\,. \tag{4.23}$$

The inner product of these states is given by the gaussian matrix model [4, 5]

$$\langle J_1', \dots, J_n' | J_1, \dots, J_m \rangle = \int dZ d\overline{Z}\, e^{-\mathrm{Tr}(Z\overline{Z})} \prod_{i=1}^{n} \mathrm{Tr}\left(\overline{Z}^{J_i'}\right) \prod_{k=1}^{m} \mathrm{Tr}\left(Z^{J_k}\right). \tag{4.24}$$

This can be shown by inserting the completeness relation of coherent state

$$\mathbf{1} = \int dZ d\overline{Z}\, e^{-\mathrm{Tr}(Z\overline{Z})} |Z\rangle\langle Z| \tag{4.25}$$

where the coherent state $|Z\rangle$ is defined by

$$\langle Z| = \langle 0| \exp\left[\mathrm{Tr}(ZB_0)\right], \qquad |Z\rangle = \exp\left[\mathrm{Tr}\left(\overline{Z}B_0^\dagger\right)\right]|0\rangle. \tag{4.26}$$

For example, the inner product between the 1-trace state and the n-trace state can be evaluated as[2]

$$\langle J | J_1, \dots, J_n \rangle = \frac{1}{J+1} \sum_{\epsilon_1, \dots, \epsilon_n = \pm 1} \frac{\Gamma\left(N + \frac{1}{2}J + \frac{1}{2}\sum_{i=1}^{n} \epsilon_i J_i + 1\right)}{\Gamma\left(N - \frac{1}{2}J + \frac{1}{2}\sum_{i=1}^{n} \epsilon_i J_i\right)} \prod_{j=1}^{n} \epsilon_j. \tag{4.27}$$

In the double scaling limit (1.1), this amplitude reduces to

$$\langle J | J_1, \dots, J_n \rangle_{PP} = \frac{JN^J}{g_2} \prod_{i=1}^{n} 2 \sinh\left(\frac{g_2}{2}\alpha_i\right), \tag{4.28}$$

where α_i is defined by

$$\alpha_i = \frac{J_i}{J}, \quad \sum_{i=1}^{n} \alpha_i = 1. \tag{4.29}$$

Note that the large g_2 behavior of $\langle J | J_1, \dots, J_n \rangle_{PP}$ is independent of n

$$\lim_{g_2 \to \infty} \langle J | J_1, \dots, J_n \rangle_{PP} = \frac{JN^J}{g_2} e^{\frac{1}{2}g_2}. \tag{4.30}$$

For the general amplitude (4.24), the pp-wave limit is not so simple as (4.28).

5. Symmetry breaking and PP-wave algebra

In this section, we consider the symmetry breaking from the conformal symmetry to the symmetry of pp-wave background. On the state (4.23), the symmetry of $\mathcal{N} = 4$ SYM is broken as

$$\mathrm{SO}(4,2) \times \mathrm{SU}(4)_R \to \mathbf{R}_{\Delta - J} \times \mathrm{SO}(4) \times \mathrm{SO}(4). \tag{5.1}$$

We will see how the broken generators form the Heisenberg algebra $h(4) \oplus h(4)$. For simplicity, we only consider the contribution of scalar fields in the free YM limit $g_{\mathrm{YM}} = 0$. In the pp-wave limit (1.1), only the term which contains B_0 and $B_0^\dagger$ survives, and other terms are sub-leading in J. This is the basic mechanism for the appearance of the Heisenberg

[2] We only checked this relation up to $n = 3$. For general n, this was proved by C. Kristjansen [28].

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algebra (see [21] for the general argument). Let us see this more explicitly. The broken generator in the SO$(4,2)$ part is given by

$$\begin{aligned} M_{0a} - iM_{5a} &= \frac{e^{-it}}{2\pi^2}\mathrm{Tr}\int_{\mathbf{S}^3}\left[-in^a\left\{\frac{1}{2}(\partial_0\phi^m)^2+\frac{1}{2}(\nabla_i\phi^m)^2+(\phi^m)^2\right\}+\right.\\ &\qquad\qquad\left.+\,\partial_0\phi^m(\partial_a\phi^m-n^a\phi^m)\right]\\ &= -i\sqrt{2}e^{-it}\mathrm{Tr}\,(a_1^{a,m\dagger}a_0^m)+\cdots\\ &= -i\sqrt{2}e^{-it}\mathrm{Tr}\,(B_1^{a\dagger}B_0)+\cdots\,. \end{aligned} \tag{5.2}$$

Here we kept only the term containing B_0. The hermitian conjugate of this operator is

$$M_{0a}+iM_{5a} = i\sqrt{2}e^{it}\mathrm{Tr}\,(B_1^aB_0^\dagger)+\cdots\,. \tag{5.3}$$

Therefore, the commutation relation of these operators becomes $h(4)$:

$$[M_{0a}+iM_{5a},M_{0b}-iM_{5b}]\sim 2\delta^{ab}\mathrm{Tr}\,(B_0^\dagger B_0)\sim 2\delta^{ab}J\,. \tag{5.4}$$

Here '$\sim$' means 'equal when they act on a state with large J charge'.

The structure in the SU$(4)_R$ part is similar. The broken generator in the SU$(4)_R$ part is given by

$$J^{6s}+iJ^{9s} = i\sqrt{2}\mathrm{Tr}\sum_{\ell=0}^{\infty}(-a_\ell^sB_\ell^\dagger+a_\ell^{s\dagger}A_\ell) = -i\sqrt{2}\mathrm{Tr}\,(a_0^sB_0^\dagger)+\cdots\,, \tag{5.5}$$

and its conjugate is

$$J^{6s}-iJ^{9s} = i\sqrt{2}\mathrm{Tr}\,(a_0^{s\dagger}B_0)+\cdots\,. \tag{5.6}$$

Therefore, their commutation relation also becomes the Heisenberg algebra $h(4)$

$$[J^{6s}+iJ^{9s},J^{6t}-iJ^{9t}]\sim 2\delta^{st}\mathrm{Tr}\,(B_0^\dagger B_0)\sim 2\delta^{st}J\,. \tag{5.7}$$

We can see that the broken generators are given by the oscillators of $\widetilde{\phi}_1^a$ and ϕ_0^s dressed by B_0. When they act on the ground state $\mathrm{Tr}(B_0^{\dagger J})|0\rangle$, one of the $B_0^\dagger$ is replaced by $a_0^{s\dagger}$ or $B_1^{a\dagger}$. $H-J$ behaves as the number operator of these dressed oscillators since $H-J$ is independent of B_0. The states created by these broken generators are the Nambu-Goldstone modes with $H-J=1$ [1, 8], which correspond to the supergravity modes in the pp-wave background.

6. Discussions

In this paper, we constructed the generators of superconformal symmetry of U(N) $\mathcal{N}=4$ SYM by generalizing [18] for the U(1) gauge group. We also studied the symmetry breaking by the large R-charge state and the appearance of the pp-wave algebra. Our analysis is limited to the bosonic part of the symmetry in the free YM limit. It is interesting to study the contraction of the whole superconformal symmetries at the interacting level.

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The dual theory of the string theory on the pp-wave background will be related to the matrix model obtained by the KK reduction of $\mathcal{N}=4$ SYM on $\mathbf{S}^3$. However, the free string spectrum in the pp-wave background is not correctly reproduced if we only keep the lowest KK modes. Let us recall the argument in [1]. In the planar limit, the string of $B_0^\dagger$'s in (4.23) can be regarded as a lattice, and the interaction $\mathrm{Tr}[Z_0,\phi_0^s][\phi_0^s,\overline{Z}_0]$ becomes the spatial derivative of the effective (1+1)-dimensional field $\phi_0^s(t,\sigma)$ in the limit (1.1). We can repeat the same argument for other light fields. Let us look at the $J=\frac{1}{2}$ component ψ of λ with angular momentum $\ell=0$

$$\psi=\frac{2\pi}{g_{\mathrm{YM}}}\cdot\frac{1}{2}(1+i\Gamma^6\Gamma^9)\lambda_{\ell=0}\,. \tag{6.1}$$

The kinetic term of ψ comes from the Yukawa interaction

$$L_Y=\frac{g_{\mathrm{YM}}}{4\pi}\mathrm{Tr}\Big(\overline{\psi}\Gamma^{12}[B_0,\psi]+\overline{\psi}\Gamma^{34}[B_0^\dagger,\psi]\Big)=\frac{g_{\mathrm{YM}}}{4\pi}\mathrm{Tr}\Big(\overline{\psi}i\Gamma^9[B_0,\psi]\Big)\,, \tag{6.2}$$

where we used the fact that $\Gamma^{34}=(\Gamma^6-i\Gamma^9)/2=0$ on the $i\Gamma^6\Gamma^9=1$ subspace. On the large R-charge state $\mathrm{Tr}(B_0^{\dagger J})|0\rangle$, (6.2) can be replaced by

$$L_Y=\frac{g_{\mathrm{YM}}\sqrt{N}}{4\pi}\sum_{j=1}^{J}\overline{\psi}_j i\Gamma^9(\psi_{j+1}-\psi_j)\,. \tag{6.3}$$

By introducing the coordinate σ as

$$\sigma=\frac{1}{\sqrt{\lambda'}}\frac{2\pi}{J}j \tag{6.4}$$

and taking the limit (1.1), the effective kinetic term of ψ is found to be

$$L_{kin}=\frac{i}{2}\int_0^{\frac{2\pi}{\sqrt{\lambda'}}}d\sigma\,\overline{\psi}(\Gamma^0\partial_0+\Gamma^9\partial_\sigma)\psi\,. \tag{6.5}$$

Unfortunately, the structure of Γ-matrices in the mass term of ψ is different from the one in the worldsheet action of the pp-wave string. The hamiltonian of $\widetilde{\phi}_1^a$ obtained from the interaction $\mathrm{Tr}[Z,\overline{Z}]^2$ is also different from the expected form, if we naively reduce this term to the KK modes with $H-J\leq 1$. It is interesting to study when the reduction to the lowest KK modes is meaningful.

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A. SO(6) and SO(9,1) gamma matrices

In this appendix we summarize our notation of Γ-matrices. We follow the notation in [29] (with slight modification). The 10-dimensional Γ-matrices are defined by $\{\Gamma^M,\Gamma^N\}=2\eta^{MN}$, where $\eta^{MN}=\mathrm{diag}(-,+^9)$ and $M,N=0,\ldots,9$.

Under the dimensional reduction to 4 dimensions, the $D = 10$ Majorana-Weyl spinor is decomposed as

$$\mathrm{SO}(9,1) \supset \mathrm{SO}(3,1) \times \mathrm{SU}(4) \mathbf{16}_L = (\mathbf{2},\mathbf{4}) \oplus \left(\overline{\mathbf{2}},\overline{\mathbf{4}}\right), \tag{A.1}$$

where we identified $\mathrm{SO}(6) \simeq \mathrm{SU}(4)$. Due to the Majorana condition, $(\mathbf{2},\mathbf{4})$ and $(\overline{\mathbf{2}},\overline{\mathbf{4}})$ are charge conjugate to each other in the 4-dimensional sense. Note that $\mathbf{6}$ of SO(6) corresponds to the antisymmetric tensor of $\mathbf{4}$ in the SU(4) picture. We use $\mu,\nu = 0,\dots,3$ and $m,n = 4,\dots,9$ as the SO(3,1) and SO(6) indices, and $A,B = 1,\dots,4$ as the indices of $\mathbf{4}$.

The SO(9,1) gamma matrices can be decomposed as

$$\Gamma^\mu = \gamma^\mu \otimes 1_8\,, \qquad \Gamma^{AB} = \gamma_5 \otimes \begin{pmatrix} 0 & -\widetilde{\rho}^{AB} \\ \rho^{AB} & 0 \end{pmatrix} = -\Gamma^{BA}\,. \tag{A.2}$$

Here $\gamma_5 = i\gamma^{0123}$ and Γ^{AB} satisfies $\{\Gamma^{AB},\Gamma^{CD}\} = \epsilon^{ABCD}$. ρ^{AB} and $\widetilde{\rho}^{AB}$ are defined by

$$(\rho^{AB})_{CD} = \delta^A_C\delta^B_D - \delta^A_D\delta^B_C\,, \quad (\widetilde{\rho}^{AB})^{CD} = \frac{1}{2}\epsilon^{CDEF}(\rho^{AB})_{EF} = \epsilon^{ABCD}\,. \tag{A.3}$$

SO(6) and SU(4) basis are related as

$$\begin{aligned} X^{AB} &= \frac{1}{2}\epsilon^{ABCD}X_{CD}\,, \qquad X^{AB} = -X^{BA} = X^\dagger_{AB}\,, \qquad X_{i4} = \frac{1}{2}(X_{i+3} + iX_{i+6}) \\ \Gamma^{i4} &= \frac{1}{2}(\Gamma^{i+3} - i\Gamma^{i+6})\,, \qquad X_{AB}\Gamma^{AB} = X_m\Gamma^m\,. \end{aligned} \tag{A.4}$$

The charge conjugation matrix and the chirality matrix are given by

$$C_{10} = C_4 \otimes \begin{pmatrix} 0 & 1_4 \\ 1_4 & 0 \end{pmatrix}, \qquad \Gamma^{11} = \Gamma^{0\cdots 9} = \gamma_5 \otimes \begin{pmatrix} 1_4 & 0 \\ 0 & -1_4 \end{pmatrix}, \tag{A.5}$$

where $(\Gamma^M)^T = -C_{10}^{-1}\Gamma^M C_{10}$ and C_4 is the charge conjugation in $D = 4$.

The Majorana-Weyl spinor in $D = 10$ is now decomposed as

$$\Psi_{MW} = \Gamma^{11}\Psi_{MW} = \begin{pmatrix} \psi^A_+ \\ \psi_{-A} \end{pmatrix} \tag{A.6}$$

where ψ_{-A} is the charge conjugation of ψ^A_+

$$\psi_{-A} = (\psi^A_+)^c = C_4(\overline{\psi}_{+A})^T\,, \qquad \gamma_5\psi_\pm = \pm\psi_\pm\,. \tag{A.7}$$

B. (Conformal) Killing vectors on $\mathbf{R} \times \mathbf{S}^3$

The Killing vectors on $\mathbf{R} \times \mathbf{S}^3$

$$\begin{aligned} \xi_{31} + i\xi_{41} &= e^{i\chi}\left[\sin\psi\partial_\theta + \cot\theta\left(\cos\psi\partial_\psi + \frac{1}{\sin\psi}\partial_\chi\right)\right] \\ \xi_{32} + i\xi_{42} &= e^{i\chi}(\partial_\psi + i\cot\psi\partial_\chi) \\ \xi_{34} &= -\partial_\chi\,, \qquad \xi_{12} = -\cot\psi\partial_\theta + \cot\theta\sin\psi\partial_\psi\,, \qquad \xi_{05} = \partial_t\,. \end{aligned} \tag{B.1}$$

The conformal Killing vectors on $\mathbf{R} \times \mathbf{S}^3$

$$
\begin{aligned}
\xi_{01} - i\xi_{51} &= e^{-it}\left(-i\cos\theta\partial_t - \sin\theta\partial_\theta\right)\\
\xi_{02} - i\xi_{52} &= e^{-it}\left(-i\sin\theta\cos\psi\partial_t + \cos\theta\cos\psi\partial_\theta - \frac{\sin\psi}{\sin\theta}\partial_\psi\right)\\
\xi_{03} - i\xi_{53} &= e^{-it}\bigg(- i\sin\theta\sin\psi\cos\chi\partial_t\\
&\qquad\qquad + \cos\theta\sin\psi\cos\chi\partial_\theta + \frac{\cos\psi\cos\chi}{\sin\theta}\partial_\psi - \frac{\sin\chi}{\sin\theta\sin\psi}\partial_\chi\bigg)\\
\xi_{04} - i\xi_{54} &= e^{-it}\bigg(- i\sin\theta\sin\psi\sin\chi\partial_t\\
&\qquad\qquad + \cos\theta\sin\psi\sin\chi\partial_\theta + \frac{\cos\psi\sin\chi}{\sin\theta}\partial_\psi + \frac{\cos\chi}{\sin\theta\sin\psi}\partial_\chi\bigg)\,. \qquad \text{(B.2)}
\end{aligned}
$$

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